

SCHRÖDINGER-KIRCHHOFF-POISSON TYPE SYSTEMS

CYRIL JOEL BATKAM AND JOÃO R. SANTOS JÚNIOR

ABSTRACT. In this article we study the existence of solutions to the system

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2) \Delta u + \phi u = f(x, u) & \text{in } \Omega \\ -\Delta \phi = u^2 & \text{in } \Omega \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of \mathbb{R}^N ($N = 1, 2$ or 3), $a > 0$, $b \geq 0$, and $f : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which is 3-superlinear. By using some variants of the mountain pass theorem established in this paper, we show the existence of three solutions: one positive, one negative, and one which changes its sign. Furthermore, in case f is odd with respect to u we obtain an unbounded sequence of sign-changing solutions.

1. INTRODUCTION

In this article, we study the existence of signed and sign-changing solutions to the following Schrödinger-Kirchhoff-Poisson system

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2) \Delta u + \phi u = f(x, u) & \text{in } \Omega \\ -\Delta \phi = u^2 & \text{in } \Omega \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{SKP})$$

where Ω is a bounded smooth domain of \mathbb{R}^N with $N = 1, 2$ or 3 ; f is a continuous function satisfying some conditions we will precise later, $a > 0$ and $b \geq 0$.

When $a = 1$ and $b = 0$, (SKP) reduces to the boundary value problem

$$\begin{cases} -\Delta u + \phi u = f(x, u) & \text{in } \Omega, \\ -\Delta \phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Knowledge of the solutions of system (1.1) is relevant in the study of stationary solutions $\psi(x, t) = e^{-it}u(x)$ to the nonlinear parabolic Schrödinger-Poisson system

$$\begin{cases} -i \frac{\partial \psi}{\partial t} = -\Delta \psi + \phi(x)\psi - |\psi|^{p-2}\psi & \text{in } \Omega, \\ -\Delta \phi = |\psi|^2 & \text{in } \Omega, \\ \psi = \phi = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

The first equation in (1.2), called Schrödinger equation, describes quantum (non-relativistic) particles interacting with the electromagnetic field generated by the motion. An interesting class of Schrödinger equations is when the potential $\phi(x)$ is determined by the charge of wave function itself, that is, when the second equation

2010 *Mathematics Subject Classification.* Primary: 35J20, 35J47 ; Secondary: 35B33, 35J25, 35J50, 35J65.

Key words and phrases. Schrödinger-Kirchhoff-Poisson system, Mountain pass theorem, Variational methods, Signed solution, Sign-changing solution.

in (1.2) (Poisson equation) holds. For more details about the physical relevance of the Schrödinger-Poisson system, we refer to [5, 10, 24].

System (1.1) has been extensively studied after the seminal work of Benci and Fortunato [10]. Many important results concerning existence and non existence of solutions, multiplicity of solutions, least energy solutions, radial and non radial solutions, semiclassical limit and concentrations of solution have been obtained. See for instance [2, 5, 6, 8, 11, 12, 25] and the references therein.

On the other hand, considering just the first equation in (SKP) with the potential equal to zero, we have the problem

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u|^2) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

which represents the stationary and N-dimensional version of the Kirchhoff model [18] for small transverse vibrations of an elastic string by considering the effect of the changing in the length during the vibrations. In fact, since the length of the string is variable during the vibrations, the tension changes with the time and depends of the L^2 norm of the gradient of the displacement u . More precisely, we have $a = P_0/h$ and $b = E/2L$, where L is the length of the string, h is the area of cross-section, E is the Young modulus of the material and P_0 is the initial tension. Problem (1.3) is called nonlocal because of the presence of the term $\int_{\Omega} |\nabla u|^2 dx$ which implies, when $b \neq 0$, that the equation in (1.3) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which make the study of such class of problems particularly interesting. Some existence and multiplicity results on Kirchhoff type problems can be found in [1, 3, 4, 8, 21, 22] and the references therein.

An important fact about system (SKP) is that it can be converted into a binonlocal problem of the Schrödinger-Kirchhoff type. More precisely, by using standard arguments as those in [10], one can show that $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a weak solution of (SKP) if, and only if, $\phi = \phi_u$ and u is a weak solution to the following Schrödinger-Kirchhoff type system

$$\begin{cases} - (a + b \int_{\Omega} |\nabla u|^2) \Delta u + \phi_u u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\text{SK})$$

where ϕ_u is the unique element of $H_0^1(\Omega)$, given by the Lax Milgram Theorem, such that $-\Delta \phi_u = u^2$. We will then concentrate our efforts in the study of (SK).

In recent years, Schrödinger-Kirchhoff problems like (SK), with $\phi = \phi(x)$ depending only on x , have received great attention of the mathematical community. In [16], by using Lusternik-Schnirelmann theory and minimax methods, He and Zou proved a result of multiplicity and concentration behavior of positive solutions for the following equation

$$\begin{cases} - (\varepsilon^2 a + b \varepsilon \int_{\mathbb{R}^3} |\nabla u|^2) \Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^3 \\ u \in H^1(\mathbb{R}^3), \end{cases} \quad (1.4)$$

by assuming, among others, that $f \in C^1(\mathbb{R}^3)$ is subcritical and 3-superlinear, and that the potential V satisfies:

$$V_{\infty} = \liminf_{|x| \rightarrow \infty} V(x) > V_0 = \inf_{\mathbb{R}^3} V(x) > 0.$$

In [26], Wang et al. replaced the second member of (1.4) by $\lambda f(u) + u^5$ and they obtained, assuming only that f is continuous, multiple positive solutions

when the parameter $\lambda > 0$ is large enough. Their approach combines Lusternik-Schnirelmann theory, minimax methods, and the Nehari manifold method. More recently, Figueiredo and Santos Júnior [15] obtained, using penalization method and the Nehari manifold approach, a multiplicity and concentration result of positive solutions for the problem

$$\begin{cases} -M\left(\frac{1}{\varepsilon}\int_{\mathbb{R}^3}|\nabla u|^2 + \frac{1}{\varepsilon^3}\int_{\mathbb{R}^3}V(x)u^2\right)[- \varepsilon^2\Delta u + V(x)u] = f(u) & \text{in } \mathbb{R}^3, \\ u \in H_1^1(\mathbb{R}^3), \end{cases}$$

with f 3-superlinear and only continuous, M a general continuous function, and V satisfying the condition: *for each $\delta > 0$ there is a bounded and Lipschitz domain $\Omega \subset \mathbb{R}^3$ such that $V_0 < \min_{\partial\Omega} V$, with $\Pi = \{x \in \Omega : V(x) = V_0\} \neq \emptyset$ and $\Pi_\delta = \{x \in \mathbb{R}^3 : \text{dist}(x, \Pi) \leq \delta\} \subset \Omega$.* We also refer to [14, 17, 23, 28] for related results.

Motivated by the previous work, we study the existence of solutions to the system (SKP) or, equivalently, to the system (SK). As far as we know, this is the first paper to investigate a bi-nonlocal problem of this type. We emphasize that the combined effects of the two nonlocal terms it contains make problem (SK) an interesting variational problem. We will have to circumvent some new difficulties in order to decide the sign of the solutions. To enunciate our main result, we first require some conditions on the nonlinear term f :

(f_1) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exists a constant $c > 0$ such that

$$|f(x, t)| \leq c(1 + |t|^{p-1}), \quad \text{where } 4 < p < 6;$$

(f_2) $f(x, t) = o(|t|)$, uniformly in $x \in \overline{\Omega}$, as $u \rightarrow 0$;

(f_3) there exists $\mu > 4$ such that $0 < \mu F(x, t) \leq tf(x, t)$ for all $t \neq 0$ and for all $x \in \overline{\Omega}$, where $F(x, t) = \int_0^t f(x, s)ds$.

We say that the couple (u, ϕ) is a sign-changing solution of (SKP) if u changes its sign. Our main result reads as follows:

Main Theorem. *Let $a > 0$ and $b \geq 0$. If $(f_1) - (f_3)$ hold, then problem (SKP) has at least three solutions: one positive, one negative, and one sign-changing. If moreover f is odd with respect to its second variable, then problem (SKP) has infinitely many sign-changing solutions.*

Our approach in proving this theorem is variational and relies on the application of three critical point theorems. The first one is a new version of the mountain pass theorem established in this paper. More precisely, using the quantitative deformation lemma introduced in [8], we derive a variant of the mountain pass theorem on cones which yields positive and negative Palais-Smale sequences for the energy functional associated to (SK). The second critical point theorem is a sign-changing version of the mountain pass theorem, also established in this paper, which guarantees the existence of a sign-changing solution of mountain pass type to (SK). The main feature of this result is a new characterization of the mountain pass level introduced recently in [19]. However, we point out that the critical point theorem in [19] cannot be used in our situation because the auxiliary operator constructed in Section 3 is not compact if it is defined on an infinite dimensional vector space. Finally, the third critical point theorem is a version of the symmetric mountain pass theorem, established recently in [8], which will be used, in case f is odd in u , to ensure the existence of infinitely many high energy sign-changing solutions to (SK).

The paper is organized as follows: In Section 2, we state and prove the abstract results. In Section 3, we provide the proof of Theorem 1, which is divided into three parts.

2. CRITICAL POINT THEOREMS

In this section, we provide some critical point theorems which are interesting by themselves and can be used in many other situations.

Let J be a C^1 -functional defined on a Hilbert space X of the form

$$X := \overline{\bigoplus_{j=1}^{\infty} X_j}, \quad \text{with } \dim X_j < \infty. \quad (2.1)$$

We introduce for $m > 2$ the following notations:

$$Y_m := \bigoplus_{j=1}^m X_j, \quad J_m := J|_{Y_m},$$

$$K_m := \{u \in Y_m ; J'_m(u) = 0\} \quad \text{and} \quad E_m := Y_m \setminus K_m.$$

Let P_m be a closed convex cone of Y_m . We set for $\mu_m > 0$

$$\pm D_m^0 := \{u \in Y_m \mid \text{dist}(u, \pm P_m) < \mu_m\}.$$

We will also denote the α -neighborhood of $W \subset Y_m$ by

$$V_\alpha(W) := \{u \in Y_m \mid \text{dist}(u, W) \leq \alpha\}, \quad \forall \alpha > 0.$$

We consider the following situation:

(A_0) There exists a locally Lipschitz continuous vector field $B : E_m \rightarrow Y_m$ (B odd if J is even) such that:

- (i) $B((\pm D_m^0) \cap E_m) \subset \pm D_m^0$;
- (ii) there exists a constant $\alpha_1 > 0$ such that $\langle J'_m(u), u - B(u) \rangle \geq \alpha_1 \|u - B(u)\|^2$, for any $u \in E_m$;
- (iii) for $\rho_1 < \rho_2$ and $\alpha > 0$, there exists $\beta > 0$ such that $\|u - B(u)\| \geq \beta$ if $u \in Y_m$ is such that $J_m(u) \in [\rho_1, \rho_2]$ and $\|J'_m(u)\| \geq \alpha$.

We have the following quantitative deformation lemma.

Lemma 2.1 (see [8], Lemma 2.3). *Let $J \in C^1(X, \mathbb{R})$. Assume that for all $m > 2$ there exists $\mu_m > 0$ such that the condition (A_0) is satisfied. Let $c \in \mathbb{R}$, $\varepsilon_0 > 0$ and $W \subset Y_m$ (with $-W = W$ if J is even) such that*

$$\forall u \in J_m^{-1}([c - 2\varepsilon_0, c + 2\varepsilon_0]) \cap V_{\frac{\mu_m}{2}}(W) : \|J'_m(u)\| \geq \varepsilon_0. \quad (2.2)$$

Then for some $\varepsilon \in]0, \varepsilon_0[$ there exists $\eta \in C([0, 1] \times Y_m, Y_m)$ such that:

- (i) $\eta(t, u) = u$ for $t = 0$ or $u \notin J_m^{-1}([c - 2\varepsilon, c + 2\varepsilon])$;
- (ii) $\eta(1, J_m^{-1}([- \infty, c + \varepsilon]) \cap W) \subset J_m^{-1}([- \infty, c - \varepsilon])$;
- (iii) $J_m(\eta(\cdot, u))$ is not increasing, for any u ;
- (iv) $\eta([0, 1] \times (\pm D_m^0)) \subset \pm D_m^0$;
- (v) If J is even then $\eta(t, \cdot)$ is odd, for any $t \in [0, 1]$.

To find positive and negative solutions we will use the following version of the mountain pass theorem in cones.

Theorem 2.2. *Let $J \in C^1(X, \mathbb{R})$. Assume that for any $m > 2$ there exists $\mu_m > 0$ such that (A_0) is satisfied. Assume also that there exist $e^\pm \in \pm P_2$ and $r > 0$ such that*

$$(A_1) \quad \|e^\pm\| > r \quad \text{and} \quad \rho := \inf_{\substack{u \in X \\ \|u\|=r}} J(u) > \delta := \max\{J(0), J(e^\pm)\}.$$

Then there exist sequences $\{u_{m,n}^\pm\}_n \subset \overline{\pm D_m^0}$ such that

$$\lim_{n \rightarrow \infty} J'_m(u_{m,n}^\pm) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} J(u_{m,n}^\pm) \in [\rho, \max_{t \in [0,1]} J(te^\pm)].$$

Proof. We define

$$c_m^\pm := \inf_{\gamma \in \Gamma_m^\pm} \sup_{u \in \gamma([0,1])} J(u),$$

where

$$\Gamma_m^\pm := \left\{ \gamma \in C([0,1], \overline{\pm D_m^0}) ; \gamma(0) = 0, \gamma(1) = e^\pm \right\}.$$

One can verify easily that the map $\gamma : [0,1] \rightarrow \overline{\pm D_m^0}$ defined by $\gamma(t) = te^\pm$ belongs to Γ_m^\pm .

By remarking that $\inf_{\substack{u \in \overline{\pm D_m^0} \\ \|u\|=r}} J(u) \geq \inf_{\substack{u \in X \\ \|u\|=r}} J(u)$, we deduce from (A_1) that $c_m^\pm \geq \rho$.

We claim that

$$\forall \varepsilon_0 \in]0, (c_m^\pm - \delta)[, \exists u \in J^{-1}([c_m^\pm - 2\varepsilon_0, c_m^\pm + 2\varepsilon_0]) \cap (\overline{\pm D_m^0}) ; \|J'_m(u)\| < \varepsilon_0. \quad (2.3)$$

Indeed, if the claim is not true then there exists $\varepsilon_0 \in]0, (c_m^\pm - \delta)/2[$ such that $\|J'_m(u)\| \geq \varepsilon_0$ for all $u \in J^{-1}([c_m^\pm - 2\varepsilon_0, c_m^\pm + 2\varepsilon_0]) \cap (\overline{\pm D_m^0})$. We apply Lemma 2.1 with $c = c_m^\pm$ and $W = V_{\frac{\mu_m}{2}}(\pm P_m)$ and we define

$$\theta : [0,1] \rightarrow \overline{\pm D_m^0}, \quad \theta(t) := \eta(1, \gamma(t)),$$

where $\gamma \in \Gamma_m^\pm$ satisfies

$$\sup_{u \in \gamma([0,1])} J(u) \leq c_m^\pm + \varepsilon, \quad (2.4)$$

with ε and η given by Lemma 2.1.

It is not difficult to show, using the properties of η , that θ belongs to Γ_m^\pm .

Inequality (2.4) and conclusion (ii) of Lemma 2.1 imply that

$$\begin{aligned} \sup_{u \in \theta([0,1])} J(u) &= \sup_{u \in \eta(1, \gamma([0,1])) \cap \overline{\pm D_m^0}} J_m(u) \\ &\leq \sup_{u \in \eta(1, J_m^{-1}([- \infty, c_m^\pm + \varepsilon])) \cap \overline{\pm D_m^0}} J_m(u) \\ &\leq c_m^\pm - \varepsilon. \end{aligned}$$

This contradicts the definition of c_m^\pm . Consequently (2.3) holds and the conclusion of the theorem follows. \square

To find sign-changing critical points of mountain pass type, we will use a version of the mountain pass theorem with a new characterization of the mountain pass level given in [19]. We will use the following notations:

$$D_m = D_m^0 \cup (-D_m^0) \quad \text{and} \quad S_m := Y_m \setminus D_m.$$

Theorem 2.3. *Let $J \in C^1(X, \mathbb{R})$. Assume that for $m > 2$ there exists $\mu_m > 0$ such that (A_0) is satisfied. Assume also that there exists a continuous map $\varphi_0 : \Delta \rightarrow Y_m$ satisfying:*

- (1) $\varphi_0(\partial_1 \Delta) \subset D_m^0$ and $\varphi_0(\partial_2 \Delta) \subset -D_m^0$,
- (2) $\varphi_0(\partial_0 \Delta) \cap D_m^0 \cap (-D_m^0) = \emptyset$,
- (3) $c_0 := \sup_{u \in \varphi_0(\partial_0 \Delta)} J(u) < c_m^* := \inf_{u \in \partial(D_m^0) \cap \partial(-D_m^0)} J(u)$,

where

$$\Delta = \{(s, t) \in \mathbb{R}^2 : s, t \geq 0, s + t \leq 1\}$$

$$\partial_1 \Delta = \{0\} \times [0, 1], \quad \partial_2 \Delta = [0, 1] \times \{0\} \quad \text{and} \quad \partial_0 \Delta = \{(s, t) \in \Delta : s + t = 1\}.$$

Then there exists a sequence $(u_m^n)_n \subset V_{\frac{\mu_m}{2}}(S_m)$ such that

$$\lim_{n \rightarrow \infty} J'_m(u_m^n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} J(u_m^n) \in [c_0, \sup_{u \in \varphi_0(\Delta)} J(u)].$$

Before giving the proof of this theorem, we recall the following useful intersection lemma which the proof can be found in [19].

Lemma 2.4. *If $\varphi : \Delta \rightarrow Y_m$ satisfies $\varphi(\partial_1 \Delta) \subset D_m^0$, $\varphi(\partial_2 \Delta) \subset -D_m^0$, and $\varphi(\partial_0 \Delta) \cap D_m^0 \cap (-D_m^0) = \emptyset$, then $\varphi(\Delta) \cap \partial(D_m^0) \cap \partial(-D_m^0) \neq \emptyset$.*

Proof of Theorem 2.3. We define

$$\Gamma_m := \{\varphi \in C(\Delta, Y_m) : \varphi(\partial_1 \Delta) \subset D_m^0, \varphi(\partial_2 \Delta) \subset -D_m^0 \text{ and } \varphi|_{\partial_0 \Delta} = \varphi_0\}.$$

It is clear that $\varphi_0 \in \Gamma_m$. By Lemma 2.4 above we have $\varphi(\Delta) \cap \partial(D_m^0) \cap \partial(-D_m^0) \neq \emptyset$ for any $\varphi \in \Gamma_m$. This intersection property implies that

$$c_m := \inf_{\varphi \in \Gamma_m} \sup_{u \in \varphi(\Delta) \cap S_m} J(u) \geq c_m^* > c_0.$$

Let us show that

$$\forall \varepsilon_0 \in \left]0, \frac{c_m - c_0}{2}\right[, \exists u \in J_m^{-1}([c_m - 2\varepsilon_0, c_m + 2\varepsilon_0]) \cap V_{\frac{\mu_m}{2}}(S_m); \|J'_m(u)\| < \varepsilon_0. \quad (2.5)$$

Arguing toward a contradiction, we assume that (2.5) does not hold, that is there exists $\varepsilon_0 \in \left]0, \frac{c_m - c_0}{2}\right[$ such that $\|J'_m(u)\| \geq \varepsilon_0$, for all $u \in J_m^{-1}([c_m - 2\varepsilon_0, c_m + 2\varepsilon_0]) \cap V_{\frac{\mu_m}{2}}(S_m)$. We can then apply Lemma 2.1 with $c = c_m$ and $W = S_m$. Using ε and the deformation η given by Lemma 2.1, we define the map

$$\beta : \Delta \rightarrow Y_m, \quad x \mapsto \beta(x) := \eta(1, \varphi(x)),$$

where $\varphi \in \Gamma_m$ is chosen such that

$$\sup_{u \in \varphi(\Delta) \cap S_m} J(u) \leq c_m + \varepsilon. \quad (2.6)$$

Since $c_0 < c_m - 2\varepsilon$, condition (3) in Theorem 2.3 and conclusion (i) of Lemma 2.1 imply that $\varphi_0(\partial_0 \Delta) \subset J_m^{-1}(-\infty, c_m - 2\varepsilon]$. It follows that $\beta \in \Gamma_m$.

Now, using the conclusions (ii) and (iv) of Lemma 2.1 and the relation (2.6), one can easily verify that

$$\eta(1, \varphi(\Delta)) \cap S_m \subset \eta(1, J_m^{-1}(-\infty, c_m + \varepsilon]) \cap S_m \subset J_m^{-1}(-\infty, c_m - \varepsilon).$$

It follows that

$$\sup_{u \in \beta(\Delta) \cap S_m} J(u) = \sup_{u \in \eta(1, \varphi(\Delta)) \cap S_m} J(u) \leq c_m - \varepsilon,$$

which is in contradiction with the definition of c_m .

The above contradiction assures that (2.5) holds. We then conclude by letting $\varepsilon_0 \rightarrow 0$. \square

We terminate this section by recalling a version of the symmetric mountain pass theorem we will apply in order to get infinitely many sign-changing critical points. We introduce for $k \geq 2$ and $m > k + 2$ the following notations:

$$Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \quad Z_k^m = \bigoplus_{j=k}^m X_j, \quad B_k := \{u \in Y_k; \|u\| \leq \rho_k\},$$

$$N_k := \{u \in Z_k; \|u\| = r_k\}, \quad N_k^m := \{u \in Z_k^m; \|u\| = r_k\}, \quad \text{where } 0 < r_k < \rho_k.$$

The following result was established in [8].

Theorem 2.5. *Let $J \in C^1(X, \mathbb{R})$ be even. Assume that for $k \geq 2$ and $m > k + 2$, there exist $0 < r_k < \rho_k$ and $\mu_m > 0$ such that (A_0) and the following two conditions are satisfied:*

$$(A'_1) \quad a_k := \max_{u \in \partial B_k} J(u) < b_k := \inf_{u \in N_k} J(u).$$

$$(A'_2) \quad N_k^m \subset S_m.$$

Then there exists a sequence $(u_{k,m}^n)_n \subset V_{\frac{\mu_m}{2}}(S_m)$ such that

$$\lim_{n \rightarrow \infty} J'_m(u_{k,m}^n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} J(u_{k,m}^n) \in [b_k, \max_{u \in B_k} J(u)].$$

3. PROOF OF THE MAIN RESULT

In this section, we apply the previous abstract theorems to prove our main result. We assume throughout this section that $(f_{1,2,3})$ are satisfied. We will also denote by $|\cdot|_q$ the usual norm of the Lebesgue space $L^q(\Omega)$.

We define X to be the usual Sobolev space $H_0^1(\Omega)$ endowed with the inner product

$$\langle u, v \rangle = \int_{\Omega} \nabla u \nabla v dx$$

and norm $\|u\|^2 = \langle u, u \rangle$, for $u, v \in H_0^1(\Omega)$.

The following result is well known (see e.g [13, 24, 29]).

Lemma 3.1. *For each $u \in H_0^1(\Omega)$, there exists a unique element $\phi_u \in H_0^1(\Omega)$ such that $-\Delta \phi_u = u^2$. Moreover, ϕ_u has the following properties:*

(i) *there exists $C > 0$ such that $\|\phi_u\| \leq C\|u\|^2$ and*

$$\int_{\Omega} |\nabla \phi_u|^2 dx = \int_{\Omega} \phi_u u^2 dx \leq C\|u\|^4;$$

(ii) *$\phi_u \geq 0$ and $\phi_{tu} = t^2 \phi_u$, $\forall t > 0$;*

(iii) *if $u_n \rightarrow u$ in $H_0^1(\Omega)$, then $\phi_{u_n} \rightarrow \phi_u$ in $H_0^1(\Omega)$ and*

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi_{u_n} u_n^2 dx = \int_{\Omega} \phi_u u^2 dx.$$

We mean by a weak solution of (SK), a function $u \in H_0^1(\Omega)$ such that

$$(a + b\|u\|^2) \int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} \phi_u u v dx = \int_{\Omega} f(x, u) v dx, \quad \forall v \in H_0^1(\Omega).$$

Let us consider the functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{1}{4} \int_{\Omega} \phi_u u^2 dx - \int_{\Omega} F(x, u) dx, \quad u \in H_0^1(\Omega).$$

It is standard to verify that if (f_1) holds then J belongs to $C^1(X, \mathbb{R})$ with

$$\langle J'(u), v \rangle = (a + b\|u\|^2) \int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} \phi_u u v dx - \int_{\Omega} f(x, u) v dx, \quad \forall u, v \in H_0^1(\Omega).$$

Clearly, critical points of J are weak solutions of problem (SK).

Let us denote by $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ the distinct eigenvalues of the problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

It is well known that each λ_j has finite multiplicity, the principal eigenvalue λ_1 is simple with positive eigenfunction e_1 , and the eigenfunctions e_j corresponding to λ_j ($j \geq 2$) are sign-changing. Let X_j be the eigenspace associated to λ_j . We set for $m > 2$

$$\begin{aligned} Y_m &= \bigoplus_{j=1}^m X_j, \quad K_m := J|_{Y_m}, \quad K_m := \{u \in Y_m ; J'_m(u) = 0\} \quad \text{and} \quad E_m := Y_m \setminus K_m \\ &+ P_m := \{u \in Y_m ; u(x) \geq 0\}, \quad Z_k^m := \bigoplus_{j=k}^m X_j \quad \text{and} \quad N_k^m := \{u \in Z_k^m ; \|u\| = r_k\}. \end{aligned}$$

Note that, for all $u \in +P_m \setminus \{0\}$, we have $\int_{\Omega} ue_1 dx > 0$, while for all $u \in Z_k$, $\int_{\Omega} ue_1 dx = 0$, where e_1 denotes the first eigenfunction of the Laplace operator with Dirichlet condition. This implies that $+P_m \cap Z_k = \{0\}$. In an analogue way we can show that $-P_m \cap Z_k = \{0\}$, where $-P_m := \{u \in Y_m ; u(x) \leq 0\}$. Therefore, since N_k^m is compact, we conclude that

$$\delta_m := \text{dist}(N_k^m, -P_m \cup +P_m) > 0. \quad (3.1)$$

Now we construct an odd locally Lipschitz continuous vector field satisfying the condition (A_0) . The idea goes back to [20].

For $u \in Y_m$ fixed, we consider the functional

$$I_u(v) = \frac{1}{2} (a + b\|u\|^2) \|v\|^2 + \frac{1}{2} \int_{\Omega} \phi_u v^2 dx - \int_{\Omega} v f(x, u) dx, \quad v \in Y_m.$$

It is not difficult to see that I_u is of class C^1 , coercive, bounded from below, weakly lower semicontinuous and strictly convex. Therefore I_u admits a unique minimizer $v = Au \in Y_m$, which is the unique solution to the problem

$$-(a + b\|u\|^2) \Delta v + \phi_u v = f(x, u), \quad v \in Y_m.$$

Clearly, the set of fixed points of A coincides with K_m . Moreover the operator $A : Y_m \rightarrow Y_m$ has the following properties.

Lemma 3.2.

- (1) *A is continuous and maps bounded sets to bounded sets.*
- (2) *For any $u \in Y_m$, we have*

$$\langle J'_m(u), u - Au \rangle \geq \|u - Au\|^2, \quad (3.2)$$

$$\|J'_m(u)\| \leq (a + C\|u\|^2) \|u - Au\|. \quad (3.3)$$

- (3) *There exists $\mu_m \in (0, \delta_m)$ such that $A(\pm D_m^0) \subset \pm D_m^0$, where δ_m is defined by (3.1).*

Proof. (1) Let $(u_n) \subset Y_m$ be such that $u_n \rightarrow u$ in Y_m . We set $v_n = Au_n$ and $v = Au$. Using the definition of A , we obtain for any $w \in Y_m$

$$(a + b\|u_n\|^2) \int_{\Omega} \nabla v_n \nabla w dx + \int_{\Omega} \phi_{u_n} v_n w dx = \int_{\Omega} w f(x, u_n) dx, \quad (3.4)$$

$$(a + b\|u\|^2) \int_{\Omega} \nabla v \nabla w dx + \int_{\Omega} \phi_u v w dx = \int_{\Omega} w f(x, u) dx. \quad (3.5)$$

By choosing $w = v_n - v$ in (3.4) and in (3.5), we obtain

$$\begin{aligned} (a + b\|u_n\|^2) \|v_n - v\|^2 &= b (\|u_n\|^2 - \|u\|^2) \int_{\Omega} \nabla v \nabla (v_n - v) dx \\ &\quad + \int_{\Omega} (\phi_u v - \phi_{u_n} v_n) (v_n - v) dx + \int_{\Omega} (v_n - v) (f(x, u_n) - f(x, u)) dx. \end{aligned}$$

Observing that

$$\begin{aligned} (\phi_u v - \phi_{u_n} v_n) (v_n - v) &= -\phi_{u_n} (v_n - v)^2 + (\phi_{u_n} - \phi_u) v (v - v_n) \\ &\leq (\phi_{u_n} - \phi_u) v (v - v_n), \end{aligned}$$

we conclude, from Hölder inequality and Sobolev embedding theorem, that

$$\begin{aligned} (a + b\|u_n\|^2) \|v_n - v\|^2 &\leq b (\|u_n\|^2 - \|u\|^2) \|v\| \|v_n - v\| + c_1 |\phi_{u_n} - \phi_u|_3 \|v\| \|v_n - v\| \\ &\quad + c_2 |f(., u_n) - f(., u)|_{\frac{p}{p-1}} \|v_n - v\| \end{aligned}$$

where c_1 and c_2 are positive constants. From (f_1) and Theorem A.2 in [27], we have $f(., u_n) - f(., u) \rightarrow 0$ in $L^{\frac{p}{p-1}}(\Omega)$. On the other hand, from definition of ϕ_{u_n} and ϕ_u , we obtain $\phi_{u_n} - \phi_u \rightarrow 0$ in $L^3(\Omega)$. Hence, $v_n \rightarrow v$ in Y_m , showing that A is continuous.

To finish this item, we observe that taking $v = w = Au$ in (3.5) leads to

$$(a + b\|u\|^2) \|Au\|^2 + \int_{\Omega} \phi_u (Au)^2 dx = \int_{\Omega} Au f(x, u) dx,$$

which implies, using (f_1) and the Sobolev embedding theorem, that

$$a \|Au\| \leq c_2 (1 + \|u\|^p),$$

where c_2 is a positive constant. Therefore, Au is bounded whenever u is bounded.

(2) Taking $w = u - Au$ in (3.5), it follows that

$$(a + b\|u\|^2) \int_{\Omega} \nabla Au \nabla (u - Au) dx + \int_{\Omega} \phi_u Au (u - Au) dx = \int_{\Omega} (u - Au) f(x, u) dx,$$

whence

$$\langle J'_m(u), u - Au \rangle = (a + b\|u\|^2) \|u - Au\|^2 + \int_{\Omega} \phi_u (u - Au)^2 dx \geq a \|u - Au\|^2.$$

Moreover, using again (3.5), we have

$$\begin{aligned} \langle J'_m(u), w \rangle &= (a + b\|u\|^2) \int_{\Omega} \nabla u \nabla w dx + \int_{\Omega} \phi_u u w dx - \int_{\Omega} w f(x, u) dx \\ &= (a + b\|u\|^2) \int_{\Omega} \nabla (u - Au) \nabla w dx + \int_{\Omega} \phi_u (u - Au) w dx. \end{aligned}$$

Applying Hölder inequality and Sobolev embedding theorem, we conclude that

$$\|J'_m(u)\| \leq (a + C\|u\|^2) \|u - Au\|,$$

for some constant $C > b$.

(3) From (f_1) and (f_2) , for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|f(x, t)| \leq \varepsilon |t| + c_\varepsilon |t|^{p-1}, \quad \forall t \in \mathbb{R}. \quad (3.6)$$

Let $u \in Y_m$ and $v = Au$. We denote $w^+ = \max\{0, w\}$ and $w^- = \min\{0, w\}$, for any $w \in X$. Taking $w = v^+$ in (3.5) and using Hölder inequality, we obtain

$$\begin{aligned} (a + b\|u\|^2) \|v^+\|^2 + \int_{\Omega} \phi_u(v^+)^2 dx &= \int_{\Omega} v^+ f(x, u) dx \\ &\leq \varepsilon |u^+|_2 |v^+|_2 + c_{\varepsilon} |u^+|_p^{p-1} |v^+|_p, \end{aligned}$$

which implies

$$\|v^+\|^2 \leq \frac{1}{a} (\varepsilon |u^+|_2 |v^+|_2 + c_{\varepsilon} |u^+|_p^{p-1} |v^+|_p). \quad (3.7)$$

Since $|z^+|_s \leq |z-w|_s$, for all $z \in X$, $w \in -P$, and $1 \leq s \leq 2^*$, there exists a positive constant $c_1 = c_1(s)$ such that $|u^+|_s \leq c_1 \text{dist}(u, -P) \leq c_1 \text{dist}(u, -P_m)$. Note that $\text{dist}(v, -P_m) \leq \|v^+\|$ and, consequently, by (3.7) and Sobolev embedding theorem that

$$\text{dist}(v, -P_m) \|v^+\| \leq \|v^+\|^2 \leq c_2 (\varepsilon \text{dist}(u, -P_m) + c_{\varepsilon} \text{dist}(u, -P_m)^{p-1}),$$

where c_2 is a positive constant. Therefore,

$$\text{dist}(v, -P_m) \leq c_2 (\varepsilon \text{dist}(u, -P_m) + c_{\varepsilon} \text{dist}(u, -P_m)^{p-1}).$$

In the same way, we can prove that

$$\text{dist}(v, +P_m) \leq c'_2 (\varepsilon \text{dist}(u, +P_m) + c_{\varepsilon} \text{dist}(u, +P_m)^{p-1}).$$

where c'_2 is also a positive constant.

Hence

$$\text{dist}(v, \pm P_m) \leq c_3 (\varepsilon \text{dist}(u, \pm P_m) + c_{\varepsilon} \text{dist}(u, \pm P_m)^{p-1}),$$

where $c_3 = \max\{c_2, c'_2\}$. Fixing ε small enough, we can choose μ_m such that

$$0 < \mu_m < \min \left\{ \delta_m, \frac{1}{m} \right\}, \quad (3.8)$$

and

$$\text{dist}(v, \pm P_m) \leq \frac{1}{2} \text{dist}(u, \pm P_m) \quad \text{whenever} \quad \text{dist}(u, \pm P_m) < \mu_m.$$

It then follows that $A(\pm D_m^0) \subset \pm D_m^0$. \square

Using μ_m as above, we define

$$\pm D_m^0 := \{u \in Y_m : \text{dist}(u, \pm P_m) < \mu_m\}, \quad (3.9)$$

$$D_m = D_m^0 \cup (-D_m^0) \quad \text{and} \quad S_m := Y_m \setminus D_m.$$

Observe that we can not yet insure that the vector field $A : Y_m \rightarrow Y_m$ is locally Lipschitz continuous. However, as pointed out in [20], we can argue as in [7] and use A to construct another vector field which satisfies the condition (A_0) . More precisely, we have the following result.

Lemma 3.3. *There exists a locally Lipschitz continuous operator $B : E_m \rightarrow Y_m$ such that*

- (1) $\langle J'(u), u - Bu \rangle \geq \frac{1}{2} \|u - Au\|^2$, for any $u \in E_m$.
- (2) $\frac{1}{2} \|u - Bu\| \leq \|u - Au\| \leq 2 \|u - Bu\|$, for any $u \in E_m$.
- (3) $B((\pm D_m^0) \cap E_m) \subset \pm D_m^0$.

Moreover, if A is odd then so is B .

Proof. Follows the same steps from [7], see also [8].

Remark 1. Lemmas 3.2 and 3.3 imply that

$$\langle J'(u), u - Bu \rangle \geq \frac{1}{8} \|u - Au\|^2 \quad \text{and} \quad \|J'_m(u)\| \leq 2(a + C\|u\|^2) \|u - Au\|.$$

Lemma 3.4. *Let $\rho_1 < \rho_2$ and $\alpha > 0$. Then there exists $\beta > 0$ such that $\|u - B(u)\| \geq \beta$ if $u \in Y_m$ is such that $J_m(u) \in [\rho_1, \rho_2]$ and $\|J'_m(u)\| \geq \alpha$.*

Proof. From the definition of operator A , it follows that for each $u \in Y_m$

$$(a + b\|u\|^2) \int_{\Omega} \nabla u \nabla u dx + \int_{\Omega} \phi_u u A u dx = \int_{\Omega} u f(x, u) dx.$$

Whence,

$$\begin{aligned} J_m(u) - \frac{1}{\mu} (a + b\|u\|^2) \int_{\Omega} \nabla u \nabla (u - Au) dx - \frac{1}{\mu} \int_{\Omega} \phi_u u (u - Au) dx = \\ a \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2 + b \left(\frac{1}{4} - \frac{1}{\mu} \right) \|u\|^4 + \left(\frac{1}{4} - \frac{1}{\mu} \right) \int_{\Omega} \phi_u u^2 dx \\ + \int_{\Omega} \left(\frac{1}{\mu} f(x, u) u - F(x, u) \right) dx. \end{aligned} \quad (3.10)$$

Using (f_3) and Lemma 3.3(2), we have

$$\begin{aligned} a \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u\|^2 + b \left(\frac{1}{4} - \frac{1}{\mu} \right) \|u\|^4 &\leq |J_m(u)| + \frac{1}{\mu} (a + b\|u\|^2) \|u\| \|u - Au\| \\ &\leq |J_m(u)| + \frac{2}{\mu} (a + b\|u\|^2) \|u\| \|u - Bu\|. \end{aligned} \quad (3.11)$$

Suppose that there exists a sequence $(u_n) \subset Y_m$ such that $J_m(u_n) \in [\rho_1, \rho_2]$, $\|J'_m(u_n)\| \geq \alpha$ and $\|u_n - Bu_n\| \rightarrow 0$. From (3.11), we conclude that (u_n) is bounded. Finally, by Remark 1 we derive $J'_m(u_n) \rightarrow 0$, which is a contradiction. \square

3.1. Existence of constant sign solutions. In this subsection, we prove the existence of a positive solution and a negative solution to (SK).

The following lemma shows that J_m has the mountain pass geometry.

Lemma 3.5. *There are $e^{\pm} \in \pm P_2$ and $r > 0$ such that the condition (A_1) in Theorem 2.2 holds.*

Proof. Let $\sigma := \inf\{|\nabla u|_2^2/|u|_2^2; u \in H_0^1(\Omega)\}$. For $\varepsilon = a\sigma/2$ in (3.6), there exists $c_1 > 0$ such that

$$|F(x, t)| \leq \frac{a\sigma}{4} |t|^2 + \frac{c_1}{p} |t|^p, \quad \forall t \in \mathbb{R}.$$

The last inequality together with Lemma 3.1(ii) and Sobolev embedding theorem gives us

$$J(u) \geq \frac{a}{4} \|u\|^2 - \frac{c_2}{p} \|u\|^p,$$

where $c_2 > 0$ is constant. Thus, by choosing $r = (a/2c_2)^{\frac{1}{p-2}} > 0$, we obtain

$$J(u) \geq \left(\frac{1}{2} - \frac{1}{p} \right) \left(\frac{a}{2} \right)^{p/(p-2)} \frac{1}{c_2^{2/(p-2)}} =: c_*,$$

for all $u \in H_0^1(\Omega)$ with $\|u\| = r$.

On the other hand, from (f_3) there are positive constants c_3 and c_4 such that

$$F(x, t) \geq c_3 |t|^\mu - c_4.$$

Hence, fixing $e_*^\pm \in \pm P_2 \setminus \{0\}$, it follows from Lemma 3.1-(ii) that

$$J(te_*^\pm) \leq \left(\frac{a}{2} \|e_*^\pm\|^2\right) t^2 + \left(\frac{b}{4} \|e_*^\pm\|^4 + \frac{1}{4} \int_{\Omega} \phi_{e_*^\pm}(e_*^\pm)^2\right) t^4 - (c_3 |e_*^\pm|_\mu^\mu) t^\mu + c_4 |\Omega|,$$

for all $t > 0$. Since $\mu > 4$, we can choose $t_* > 0$ large enough such that, defining $e^\pm := t_* e_*^\pm$, we have $J(e^\pm) < 0$ and $\|e^\pm\| > r$. This shows that

$$\rho = \inf_{\substack{u \in X \\ \|u\|=r}} J(u) \geq c_* > 0 = \max\{J(0), J(e^\pm)\}.$$

□

Proof of Main Theorem (Part 1). By Lemmas 3.5, 3.3 and 3.4, and Remark 1 the conditions (A_0) and (A_1) of Theorem 2.2 are satisfied.

Applying Theorem 2.2 we find sequences $\{u_{m,n}^\pm\}_n \subset \overline{\pm D_m^0}$ such that

$$\lim_{n \rightarrow \infty} J'_m(u_{m,n}^\pm) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} J(u_{m,n}^\pm) \in [\rho, \max_{t \in [0,1]} J(te^\pm)].$$

For any $u \in Y_m$ we have, in view of (f_3) , Lemma 3.1(ii) and since $\mu > 4$,

$$J(u) - \frac{1}{\mu} \langle J'_m(u), u \rangle \geq a \left(\frac{1}{2} - \frac{1}{\mu}\right) \|u\|^2 + b \left(\frac{1}{4} - \frac{1}{\mu}\right) \|u\|^4. \quad (3.12)$$

It follows from (3.12) that the sequences $\{u_{m,n}^\pm\}_n$ are bounded in Y_m . Since Y_m is of finite dimension we have, up to subsequences, $u_{m,n}^\pm \rightarrow u_m^\pm$ in Y_m as $n \rightarrow \infty$. Since $\overline{\pm D_m^0}$ is closed and J_m is smooth, it follows that

$$u_m^\pm \in \overline{\pm D_m^0}, \quad J'_m(u_m^\pm) = 0, \quad \text{and} \quad J(u_m^\pm) \in [\rho, \max_{t \in [0,1]} J(te^\pm)]$$

Inequality (3.12) above then implies that the sequences $\{u_m^\pm\}_m$ are bounded in X . Going to subsequences if necessary we can assume that $u_m^\pm \rightarrow u^\pm$ in X and $u_m^\pm \rightarrow u^\pm$ in $L^p(\Omega)$ and in $L^q(\Omega)$, as $m \rightarrow \infty$. We denote by π_m the orthogonal projection of X onto Y_m . We have

$$\begin{aligned} \langle J'_m(u_m^\pm), u_m^\pm - \pi_m u^\pm \rangle &= (a + b \|u_m^\pm\|^2) \langle u_m^\pm, u_m^\pm - \pi_m u^\pm \rangle + \int_{\Omega} \phi_{u_m^\pm} u_m^\pm (u_m^\pm - \pi_m u^\pm) dx \\ &\quad - \int_{\Omega} (u_m^\pm - \pi_m u^\pm) f(x, u_m^\pm) dx. \end{aligned} \quad (3.13)$$

The Hölder inequality gives

$$\begin{aligned} \left| \int_{\Omega} (u_m^\pm - \pi_m u^\pm) f(x, u_m^\pm) dx \right| &\leq |u_m^\pm - \pi_m u^\pm|_p |f(x, u_m^\pm)|_{\frac{p}{p-1}} \\ \left| \int_{\Omega} \phi_{u_m^\pm} u_m^\pm (u_m^\pm - \pi_m u^\pm) dx \right| &\leq |\phi_{u_m^\pm}|_3 |u_m^\pm|_3 |u_m^\pm - \pi_m u^\pm|_3. \end{aligned}$$

Since the sequences $\{u_m^\pm\}_m$ are bounded, we deduce from (f_1) and Lemma 3.1(i) that $\{|f(x, u_m^\pm)|_{\frac{p}{p-1}}\}_m$ and $\{|\phi_{u_m^\pm}|_3\}_m$ are bounded. One can then easily deduce from (3.13), since $J'_m(u_m^\pm) = 0$ and $\pi_m u^\pm \rightarrow u^\pm$ in X , that $u_m^\pm \rightarrow u^\pm$ in X as $m \rightarrow \infty$. It is easy to see that u^\pm are critical points of J .

It remains to show that u^+ is positive and that u^- is negative.

We first remark that $J(u^\pm) \geq \rho > 0$ implies that $u^\pm \neq 0$.

Now, recall the definition (3.9) of $\pm D_m^0$. Since $u_m^\pm \in \overline{\pm D_m^0}$ and $\pm P_m \subset \pm P$, we have

$$dist(u_m^\pm, \pm P) \leq dist(u_m^\pm, \pm P_m) \leq \mu_m, \quad (3.14)$$

where μ_m is given by (3.8). Since $\mu_m \rightarrow 0$ as $m \rightarrow \infty$, we conclude by taking the limit $m \rightarrow \infty$ in (3.14) that $u^\pm \in \pm P$. \square

3.2. Existence of a sign-changing solution. We show in this subsection that (SK) has a sign-changing solution of mountain pass type under assumptions $(f_{1,2,3})$. The next two lemmas will be very helpful.

Lemma 3.6. *For $q \in [2, 6]$ there exists $\kappa_q > 0$ independent of μ_m such that*

$$|u|_q \leq \kappa_q \mu_m, \quad \forall u \in D_m^0 \cap (-D_m^0).$$

Proof. This follows from the fact that there exists $\kappa_q > 0$ such that for any $u \in Y_m$

$$|u^\pm|_q = \inf_{v \in \mp P_m} |u - v|_q \leq \kappa_q \inf_{v \in \mp P_m} \|u - v\| = \kappa_q dist(u, \mp P_m).$$

\square

Lemma 3.7. *For m large enough we have*

$$J(u) \geq \frac{a}{8} \mu_m^2 \quad \text{for } u \in \partial(D_m^0) \cap \partial(-D_m^0).$$

Proof. Let $u \in \partial(D_m^0) \cap \partial(-D_m^0)$. It is clear that $\|u^\pm\| \geq dist(u, \mp P_m) = \mu_m$. Taking $\varepsilon = \frac{a}{2\kappa_2^2}$ in (3.6) and using Lemma 3.6, we see that

$$J(u) \geq \frac{a}{2} \|u\|^2 - \int_{\Omega} F(x, u) dx \geq \frac{a}{4} \mu_m^2 - c \mu_m^p,$$

where $c > 0$ is a constant. We conclude by using the fact that $\mu_m \rightarrow 0$ as $m \rightarrow \infty$. \square

In the following proof we adopt the notations of Theorem 2.3.

Proof of Main Theorem (Part 2). We first follow [20] to verify the assumptions of Theorem 2.3. We define the continuous map $\varphi_0 : \Delta \rightarrow Y_m$ by $\varphi_0(s, t) = R(se_2^- + te_2^+)$ for all $(s, t) \in \Delta$, where e_2 is an eigenfunction corresponding to the second eigenvalue of the Laplacian, $e_2^- = \min\{e_2, 0\}$, $e_2^+ = \max\{e_2, 0\}$, and $R > 0$ is a constant to be determined later. Since e_2 is sign-changing, e_2^\pm are not equal to 0. Obviously, $\varphi_0(0, t) \in D_m^0$ and $\varphi_0(s, 0) \in -D_m^0$.

Now a simple computation shows that $\delta := \min\{|(1-t)e_2^- + te_2^+|_2 : t \in [0, 1]\} > 0$. Then, $|u|_2 \geq \delta R$ for $u \in \varphi_0(\partial_0 \Delta)$ and it follows from Lemma 3.6 that $\varphi_0(\partial_0 \Delta) \cap D_m^0 \cap (-D_m^0) = \emptyset$ for R large enough.

Recall that (f_3) implies $F(x, u) \geq c_1|u|^\mu - c_2$ for some positive constants c_1 and c_2 . Then we obtain using Lemma 3.1(i)

$$J(u) \leq \frac{a}{2} \|u\|^2 + C_1 \|u\|^4 - C_2 |u|_\mu^\mu + C_3,$$

where $C_{1,2,3}$ are positive constants. This inequality together with Lemma 3.7 implies that, for m and R large enough,

$$\sup_{u \in \varphi_0(\partial_0 \Delta)} J(u) < 0 < c_m^*.$$

We have then shown that the assumptions of Theorem 2.3 are satisfied. Hence, applying Theorem 2.3 we find, for m large enough, a sequence $(u_m^n)_n \subset V_{\frac{\mu_m}{2}}(S_m)$ such that

$$\lim_{n \rightarrow \infty} J'_m(u_m^n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} J(u_m^n) \in [c_0, \sup_{u \in \varphi_0(\Delta)} J(u)].$$

We can proceed as in the proof of Part 1 to show that, up to a subsequence, $u_m^n \rightarrow u_m$ in Y_m as $n \rightarrow \infty$ and that $u_m \in V_{\frac{\mu_m}{2}}(S_m)$, $J'_m(u_m) = 0$, and the sequence $(u_m)_m$ converges, up to a subsequence, to a critical point u of J as $m \rightarrow \infty$. To see that u is sign-changing, we first observe that

$$\langle J'_m(u_m), u_m^\pm \rangle = 0 \Rightarrow a\|u_m^\pm\|^2 \leq \int_{\Omega} u_m^\pm f(x, u_m^\pm).$$

We recall that (f_1) and (f_2) imply

$$\forall \varepsilon > 0, \quad \exists c_\varepsilon > 0 ; |f(x, t)| \leq \varepsilon|t| + c_\varepsilon|t|^{p-1}, \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$

We then obtain by using the Sobolev embedding theorem

$$a\|u_m^\pm\|^2 \leq \int_{\Omega} u_m^\pm f(x, u_m^\pm) \leq c(\varepsilon\|u_m^\pm\|^2 + c_\varepsilon\|u_m^\pm\|^p),$$

for some constant $c > 0$. Since u_m is sign-changing, u_m^\pm are not equal to 0. Choosing ε small enough it follows that $\|u_m^\pm\| \geq \alpha > 0$, where α does not depend on m . By passing to the limit $m \rightarrow \infty$, we conclude that u is sign-changing. \square

3.3. Existence of high energy sign-changing solutions. In this subsection, we show that (SKP) has infinitely many large energy sign-changing solutions by applying Theorem 2.5. We recall the definition of Y_k and Z_k ($k \geq 2$).

$$Y_k = \overline{\bigoplus_{j=1}^k X_j} \quad \text{and} \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j},$$

where X_j designates, as above, the eigenspace corresponding to the j th eigenvalue of the Laplacian.

Proof of Main Theorem (Part 3). Since the argument is the same as in [8, 9], we just provide a sketch here.

Using (f_1) and Lemma 3.1(ii), we obtain

$$J(u) \geq \frac{a}{2}\|u\|^2 - c_1|u|_p^p - c_2, \quad \forall u \in X,$$

where $c_1, c_2 > 0$ are constant. It then follows that for any $u \in Z_k$ such that

$$\|u\| = r_k := \left(\frac{c_1}{a} p \beta_k^p \right)^{\frac{1}{2-p}},$$

we have

$$J(u) \geq a \left(\frac{1}{2} - \frac{1}{p} \right) \left(\frac{c_1}{a} p \beta_k^p \right)^{\frac{2}{2-p}} - c_2,$$

where

$$\beta_k := \sup_{\substack{v \in Z_k \\ \|v\|=1}} |v|_p.$$

Then we obtain By Lemma 3.8 in [27], $\beta_k \rightarrow 0$ and then $r_k \rightarrow \infty$, as $k \rightarrow \infty$.

On the other hand, using the fact that Y_k is finite-dimensional one can easily verify that $J(u) \rightarrow -\infty$, as $\|u\| \rightarrow \infty$, $u \in Y_k$. Therefore, for k big enough we can choose $\rho_k > r_k$ such that the conditions of Theorem 2.5 are satisfied. We can then

proceed as in [8, 9] to show that J has a sequence $\{u_k\}$ of sign-changing critical points such that $J(u_k) \rightarrow \infty$, as $k \rightarrow \infty$. \square

Acknowledgements. CJB was supported by The Fields Institute for Research in Mathematical Sciences and The Perimeter Institute for Theoretical Physics through a Fields-Perimeter Africa Postdoctoral Fellowship.

REFERENCES

- [1] C.O. Alves, F.J.S.A. Corrêa, and T.F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, *Comput. Math. Appl.* **49** (2005) 85–93.
- [2] C.O. Alves and M.A.S. Souto, Existence of least energy nodal solution for a Schrödinger-Poisson system in bounded domains, *Z. Angew. Math. Phys.* **65** (2014) 1153–1166.
- [3] G. Anelo, A uniqueness result for a nonlocal equation of Kirchhoff equation type and some related open problem, *J. Math. Anal. Appl.* **373** (2011) 248–251.
- [4] G. Anelo, On a perturbed Dirichlet problem for a nonlocal differential equation of Kirchhoff type, *Boundary Value Problems* (2011) doi:10.1155/2011/891430.
- [5] A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, *Contemp. Math.* **10** (2008) 391–404.
- [6] A. Azzollini and A. Pomponio, Ground state solutions for the nonlinear Schrödinger-Maxwell equations, *J. Math. Anal. Appl.* **345** (2008) 90–108.
- [7] T. Bartsch and Z. Liu, *Multiple sign changing solutions of a quasilinear elliptic eigenvalue problem involving the p -Laplacian*, *Comm. Contemp. Math.* **234** (2004) 245–258.
- [8] C.J. Batkam, Multiple sign-changing solutions to a class of Kirchhoff type problems. arXiv:1501.05733.
- [9] C.J. Batkam, High energy sign-changing solutions to Schrödinger-Poisson type systems. arXiv:1501.05942.
- [10] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, *Methods Nonlinear Anal.* **11** (1998) 283–293.
- [11] G. Cerami and G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, *J. Differ. Equ.* **248** (2010) 521–543.
- [12] G.M. Coclite, A multiplicity result for the nonlinear Schrödinger-Maxwell equations, *Commun. Appl. Anal.* **7** (2003) 417–423.
- [13] T. D’Aprile and D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A* **134** (2004) 893–906.
- [14] G.M. Figueiredo, N. Ikoma, and J.R. Santos Júnior, Existence and Concentration Result for the Kirchhoff Type Equations with General Nonlinearities, *Arch. Rational Mech. Anal.* **213** (2014) 931–979.
- [15] G.M. Figueiredo and J.R. Santos Júnior, Multiplicity and concentration behavior of positive solutions for a Schrödinger-Kirchhoff type problem via penalization method, *ESAIM Control Optim. Calc. Var.* **20**(02) (2014) 389–415.
- [16] X. He and W. Zou, Existence and concentration of positive solutions for a Kirchhoff equation in \mathbb{R}^3 , *J. Differ. Equ.* **252** (2012) 1813–1834.
- [17] Y. Li, F. Li and J. Shi, Existence of a positive solution to Kirchhoff type problems without compactness conditions, *J. Differ. Equ.* **253** (2012) 2285–2294.
- [18] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.
- [19] J. Liu, X. Liu, and Z.-Q. Wang, Multiple mixed states of nodal solutions for nonlinear Schrödinger systems, *Calc. Var. Partial Differential Equations* **52** (2015) 565–586.
- [20] Z. Liu, Z.-Q. Wang, and J. Zhang, Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system. arXiv:1408.6870.
- [21] T.F. Ma and J.E.M. Rivera, Positive Solutions for a Nonlinear Nonlocal Elliptic Transmission Problem, *Applied Mathematics Letters* **16** (2003) 243–248.
- [22] T.F. Ma, Remarks on an elliptic equation of Kirchhoff type, *Nonlinear Anal.* **63** (2005) 1967–1977.
- [23] J. Nie and X. Wu, Existence and multiplicity of non-trivial solutions for Schrödinger-Kirchhoff equations with radial potential, *Nonlinear Analysis* **75** (2012) 3470–3479.

- [24] D. Ruiz , The Schrödinger-Poisson equation under the effect of a nonlinear local term, *J. Funct. Analysis*, **237** (2006) 655–674.
- [25] D. Ruiz and G. Siciliano, A note on the Schrödinger-Poisson-Slater equation on bounded domains, *Adv. Nonlinear Stud.* **8** (2008) 179–190.
- [26] J. Wang, L. Tian, J. Xu and F. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, *J. Differ. Equ.* **253** (2012) 2314–2351.
- [27] M. Willem, Minimax Theorems, *Birkhäuser*, Boston (1996).
- [28] X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in \mathbb{R}^N , *Nonlinear Analysis, RWA* **12** (2011) 1278–1287.
- [29] F. Zhao and L. Zhao, Positive solutions for Schrödinger-Poisson equations with a critical exponent, *Nonlinear Anal.* **70** (2009) 2150–2164.
- [30] W. Zou, On finding sign-changing solutions, *J. Funct. Anal.* **234** (2006) 364–419.

CYRIL JOEL BATKAM
 THE FIELDS INSTITUTE,
 TORONTO, ONTARIO, M5T 3J1, CANADA.
E-mail address: cbatkam@fields.utoronto.ca, cyrilbatkam@gmail.com

JOÃO R. SANTOS JÚNIOR
 UNIVERSIDADE FEDERAL DO PARÁ,
 FACULDADE DE MATEMÁTICA
 CEP 66075-110, BELÉM, PARÁ, BRAZIL.
E-mail address: joaojunior@ufpa.br, joaorodrigues_mat@hotmail.com